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Calculating Implied Default Rates from CDS Spreads

Introduction

Credit market investors have to assess yield against the probability of default constantly. In this regard, many tools have been developed to help investors to estimate the default probabilities. Rating agencies publish actuarial tables regularly based on the long history of default experiences. Other tools such as CreditSights' BondScore or KMV's EDF combine company financials and equity market valuations with historical default experiences to estimate default probabilities. It is important to realize that when market provides a CDS quote on a credit, market is actually providing a market-implied default probability, which differ from experience-based estimates in a fundamental way. In turbulent times fear causes markets to over-estimate default probabilities, and in good times greed causes market to under-estimate default probabilities.

In the following we shall discuss the mathematics involved to calculate market implied default probabilities from term structures of CDS curves. The market implied default probabilities calculated from the algorithm outlined below have wide applications in credit trading and risk management, ranging from calculating the present value of a CDS contract to the fair value spreads of various synthetic CDS tranches.

Notations

$B(t)$: Risk-free discount factor at time t.
$Q(t)$: The survival probability from time period $(0,t)$
c : CDS spreads
t_i : CDS payment dates. $i = 1,2,...N$. $t_0 = 0$ is the starting date, $t_N = T$ is the ending date.
R : Recovery rate

Basic Equations

The basic idea in determining the fair value of CDS spreads is that the expected present value of CDS premium payments must be equal to the expected present value of payout in the event of default. The expected present value of CDS premium payments is

$$c\sum_{i=1}^{N}(t_i - t_{i-1})B(t_i)Q(t_i) - c\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(t - t_{i-1})B(t)dQ(t)$$

The 2nd term represents the amount of payment accrued in the event that default occurs at time t between t_{i-1} and t_i (remember that $dQ(t) < 0$). When day counting conventions are considered, the CDS spread should be adjusted by the appropriate factors. This expected present value of payments should equal to the expected present value of the payout from the CDS contract in case of default:

$$(1-R)\int_0^T B(t)dQ(t)$$

Therefore, we have

$$c\sum_{i=1}^{N}(t_i - t_{i-1})B(t_i)Q(t_i) - c\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}(t - t_{i-1})B(t)dQ(t) + (1-R)\int_0^T B(t)dQ(t) = 0 \qquad (1)$$

Constant Hazard Rates

We apply equation (1) to the whole term structure of CDS curve: $\{c_j\}$ with maturity dates $\{T_j\}$. Furthermore, we assume that hazard rate over time period (T_{j-1}, T_j) is constant h_j.

Under the piece-wise constant hazard rate approximation we then have

$$Q(t) = Q(T_{j-1})\exp(-h_j(t - T_{j-1})),\ T_{j-1} \le t \le T_j \qquad (2)$$

In order to proceed further, we introduce a new set of notations. Over the time period (T_{j-1}, T_j), we denote the payment dates as $\{t_m^j\}$, where $m = 0,1,2,\ldots n_j$, with $t_0^j = T_{j-1}$ and $t_{n_j}^j = T_j$. The CDS premium covering the time $(0, T_j)$ is c_j. Equation (1) can then be re-written as

$$c_k\sum_{j=1}^{k}\sum_{m=1}^{n_j}(t_m^j - t_{m-1}^j)B(t_m^j)Q(t_m^j) - c_k\sum_{j=1}^{k}\sum_{m=1}^{n_j}\int_{t_{m-1}^j}^{t_m^j}(t - t_{m-1}^j)B(t)dQ(t) + (1-R)\sum_{j=1}^{k}\sum_{m=1}^{n_j}\int_{t_{m-1}^j}^{t_m^j}B(t)dQ(t) = 0$$

Bootstrapping

We also apply exponential extrapolation to the discount curve:

$$B(t) = B(t_{m-1}^j)\exp(-r_m^j(t - t_{m-1}^j)), \qquad t_{m-1}^j \le t \le t_m^j \tag{3}$$

Where

$$r_m^j = -\frac{\log(B(t_m^j)/B(t_{m-1}^j))}{t_m^j - t_{m-1}^j} \tag{4}$$

Combining Equations (1), (2) and (3), we have

$$\begin{aligned}
&c_k \sum_{j=1}^{k}\sum_{m=1}^{n_j}(t_m^j - t_{m-1}^j)B(t_m^j)Q(T_{j-1})\exp(-h_j(t_m^j - T_{j-1})) + \\
&c_k \sum_{j=1}^{k}\sum_{m=1}^{n_j}B(t_{m-1}^j)Q(T_{j-1})h_j\int_{t_{m-1}^j}^{t_m^j}(t - t_{m-1}^j)\exp(-r_m^j(t - t_{m-1}^j))\exp(-h_j(t - T_{j-1}))dt - \\
&(1-R)\sum_{j=1}^{k}\sum_{m=1}^{n_j}B(t_{m-1}^j)Q(T_{j-1})h_j\int_{t_{m-1}^j}^{t_m^j}\exp(-r_m^j(t - t_{m-1}^j))\exp(-h_j(t - T_{j-1}))dt = 0
\end{aligned}$$

After a change of variable $\tau = t - t_{m-1}^j$ and let $\Delta t_m^j = t_m^j - t_{m-1}^j$, we have

$$\begin{aligned}
&c_k \sum_{j=1}^{k}\sum_{m=1}^{n_j}(t_m^j - t_{m-1}^j)B(t_m^j)Q(T_{j-1})\exp(-h_j(t_m^j - T_{j-1})) + \\
&c_k \sum_{j=1}^{k}\sum_{m=1}^{n_j}B(t_{m-1}^j)Q(T_{j-1})h_j\exp(-h_j(t_{m-1}^j - T_{j-1}))\int_{t_{m-1}^j}^{t_m^j}\tau\exp(-(r_m^j + h_j)\tau)d\tau - \\
&(1-R)\sum_{j=1}^{k}\sum_{m=1}^{n_j}B(t_{m-1}^j)Q(T_{j-1})h_j\exp(-h_j(t_{m-1}^j - T_{j-1}))\int_{t_{m-1}^j}^{t_m^j}\exp(-(r_m^j + h_j)\tau)d\tau = 0
\end{aligned} \tag{5}$$

The two integrals in Equation (5) can be carried out explicitly

$$I(\gamma,\Delta) = \int_0^{\Delta}\exp(-\gamma\tau)d\tau = \frac{1-\exp(-\gamma\Delta)}{\gamma} \tag{6}$$

$$J(\gamma,\Delta) = \int_0^{\Delta}\tau\exp(-\gamma\tau)d\tau = -\frac{\partial I(\gamma,\Delta)}{\partial\gamma} = \frac{1-(1+\gamma\Delta)\exp(-\gamma\Delta)}{\gamma^2} \tag{7}$$

Equation (5) can be solved numerically along the credit curve, in much the same way as a risk free discount curve is constructed from a set of government bonds.

Limit of Instantaneous CDS

CDS uses actual/360 day-counting convention. This can easily be adjusted by multiplying a factor of 365/360 to $\{c_k\}$ in the above equations. In the limit of instantaneous CDS contract, Equation (1) reduces to

$$c\Delta t = (1-R)\Delta P$$

This gives the instantaneous hazard rate of

$$H(0) = \frac{\Delta P}{\Delta t} = \frac{c}{1-R} \tag{8}$$

Limit of Infinitesimal CDS premiums

In the limit of small CDS premiums, the second term in Equation (1) is of order $O(c^2)$ and can be dropped. Equation (1) reduces to

$$c\sum_{i=1}^{N}(t_i - t_{i-1})B(t_i) - (1-R)H\int_0^T B(t)dt = 0$$

This gives the relationship between c and H

$$c = \frac{(1-R)H\int_0^T B(t)dt}{\sum_{i=1}^{N}(t_i - t_{i-1})B(t_i)} \tag{9}$$

Equations (8) and (9) are useful in checking numerical algorithms.

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